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Central charges and $\mathrm{U}(1)_R$ symmetries in $\mathcal{N}=1$ super Yang-Mills

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Abstract: We use recent results of Intriligator and Wecht [1] to study the phase structure of $\mathcal{N}=1$ super Yang-Mills theory with gauge group $\mathrm{SU}(N_c)$, a chiral superfield in the adjoint, and N_f chiral superfields in the fundamental representation of the gauge group. Our discussion sheds new light on [1] and supports the conjecture that the central charge a decreases under RG flows and is non-negative in unitary four dimensional conformal field theories.

Keywords: Renormalization Group, Duality in Gauge Field Theories, Supersymmetric Effective Theories.

Contents

1. Introduction

Recently, K. Intriligator and B. Wecht [1] proposed a solution to an old problem, of determining the $U(1)_R$ charges of chiral operators at non-trivial fixed points of the Renormalization Group (RG) in $\mathcal{N}=1$ supersymmetric four dimensional gauge theories. The results of [1] also provide support for the conjectured "a-theorem" [2]–[11], which states[1] that the combination of 't Hooft anomalies

$$a \equiv \frac{3}{32}(3\,\mathrm{tr}\,R^3 - \mathrm{tr}\,R) \equiv \frac{3}{32}\tilde{a}\,, \tag{1.1}$$

is always positive and lower at an IR fixed point of an RG flow than at the corresponding UV fixed point. Here R is the $U(1)_R$ charge which belongs to the $\mathcal{N}=1$ superconformal multiplet at a fixed point of the RG, and the trace runs over the chiral fermions in the multiplets. We will mostly refer below to the quantity $\tilde{a}$ defined in (1.1), which differs from a by the factor $3/32$.

As we review below, the analysis of [1] leaves some open questions that need to be studied on a case by case basis, such as the range of validity of the results, and the implications for the a-theorem. In this note we apply the results of [1] to $\mathcal{N}=1$ super Yang-Mills (SYM) coupled to a single chiral superfield X in the adjoint representation

[1] The a-theorem actually refers to an anomaly associated with the stress-tensor, but in $\mathcal{N}=1$ superconformal theories it is related by supersymmetry to the anomaly associated with the R-current discussed here [7, 8].

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of the gauge group $SU(N_c)$, and N_f "flavors" of chiral superfields in the fundamental representation of the gauge group, Q^i, $\tilde{Q}_{\tilde{i}}$, $i, \tilde{i} = 1, \ldots, N_f$. This class of theories exhibits a rich pattern of RG flows, some of which are understood, but there are important open questions. In particular, it was pointed out in [8] that some of the known RG flows might lead to counter-examples to the a-theorem, depending on some detailed features of the flows that were not understood at the time.

Thus, this class of theories is a good testing ground for the techniques of Intriligator and Wecht (IW). We will see that the results of [1] allow one to obtain a more detailed picture of the phase structure. In the process, we will get new insights into the construction of [1], and the validity of the a-theorem. To set the stage, we start with a general discussion of RG flows and fixed points in $\mathcal{N} = 1$ SYM.

An asymptotically free gauge theory describes an RG flow between a free theory in the UV, where the gauge coupling vanishes, and an interacting theory in the IR, where the gauge coupling is non-zero. Moreover, it might happen that an operator that is irrelevant (in the RG sense) near the free UV fixed point becomes marginal or relevant near the IR fixed point, and leads to new deformations that allow one to flow further in the space of couplings and explore additional fixed points. The program of "solving" a gauge theory involves understanding all the fixed points that can be reached from the free UV theory this way, and then understanding the theory along the RG flows that connect the different fixed points.

At a fixed point of the RG, the theory becomes scale invariant, and in all known examples conformal. We will be interested in $\mathcal{N} = 1$ supersymmetric theories, which become superconformal at fixed points. The $\mathcal{N} = 1$ superconformal algebra contains a $U(1)_R$ charge R, which determines the scaling dimensions of chiral operators, via the relation

$$\Delta(\mathcal{O}) = \frac{3}{2} R(\mathcal{O}) \,. \tag{1.2}$$

Thus, determining the R-charges of chiral operators at fixed points of the RG is important, since it leads to a determination of their scaling dimensions at these fixed points.

One general idea that is known to be useful for identifying the $U(1)_R$ charge R is to postulate that the current J_μ that becomes part of the superconformal multiplet in the IR is conserved throughout the RG flow, and thus can be identified already in the vicinity of the (asymptotically free) UV fixed point. Strong support for this idea is provided by the form of the NSVZ β function [12, 13] of $\mathcal{N} = 1$ SYM with gauge group G and chiral superfields Φ_i in the representations r_i of the gauge group:

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi} \frac{3T(G) - \sum_i T(r_i)(1 - \gamma_i)}{1 - \frac{\alpha}{2\pi} T(G)} \,. \tag{1.3}$$

Here $\alpha = g^2/4\pi$, g is the gauge coupling, γ_i is the anomalous dimension of Φ_i, $T(r)$ is the quadratic Casimir corresponding to the representation r, $\mathrm{Tr}_r(T^a T^b) = T(r)\delta^{ab}$, and $T(G) = T(r = \text{adjoint})$. If the coupling α in the IR is sufficiently small that we do not have to worry about the denominator in (1.3), non-trivial fixed points of the RG correspond to

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a zero of the numerator,

$$3T(G) - \sum_i T(r_i)(1-\gamma_i) = 0\,. \tag{1.4}$$

At a fixed point, the scaling dimension of Φ_i is given by $\Delta_i = 1 + \frac{1}{2}\gamma_i$. Using the relation between the dimension and the R-charge, (1.2), one can rewrite the condition for a fixed point (1.4) as

$$T(G) + \sum_i T(r_i)(R_i - 1) = 0\,. \tag{1.5}$$

This equation is also the condition that the R-symmetry with $R(\Phi_i) = R_i$ be anomaly free. Thus, it is natural to postulate that the R-symmetry of the IR fixed point is one of the anomaly free R-symmetries which satisfy (1.5), and are therefore conserved in the full theory. Indeed, the trace of the stress-tensor T^μ_μ and the divergence of the $\mathrm{U}(1)_R$ current $\partial_\mu J^\mu$ are in the same supersymmetry multiplet. Thus, one expects the condition that $\beta(\alpha) = 0$ (or $T^\mu_\mu = 0$) to be related by supersymmetry to the condition that the $\mathrm{U}(1)_R$ current is anomaly free and hence conserved.

On general grounds, one does not expect that the technique described above should be always valid. It might be that the $\mathrm{U}(1)_R$ that becomes part of the superconformal algebra in the extreme IR is an accidental symmetry of the IR theory and is not visible from the UV. The argument above suggests that this does not happen when the theory is sufficiently weakly coupled, and only becomes an issue when the IR fixed point is "too far" from the UV fixed point. From the point of view of (1.3), one way this might happen is if α exceeds in the IR the value for which the β function has a pole.

There are actually two different ways in which such a violation might manifest itself:

1. Suppose that we found a candidate R-symmetry by solving (1.5), and it predicts that a particular gauge invariant chiral operator M has R-charge $R(M) < 2/3$. This is inconsistent with unitarity [14]. In that case, it is believed that what happens is the following [15]. The correct answer is $R(M) = 2/3$, M is a free field in the IR CFT, and the correct R-symmetry is a combination of the solution of (1.5) and an accidental symmetry of the IR theory which acts only on the free field M.

2. Even if the candidate R-symmetry assigns R-charge larger than 2/3 to gauge invariant chiral superfields, it may be invalid. This is in a sense more problematic, since unlike case (1), there is no obvious "smoking gun", and no general procedure for fixing the problem.

To illustrate the above general considerations, consider the case of supersymmetric QCD, with gauge group $\mathrm{SU}(N_c)$ and N_f fundamental multiplets, Q^i, $\tilde{Q}_{\tilde{i}}$. The theory is asymptotically free for $N_f < 3N_c$. Following the logic outlined above, one looks for the R-symmetry in the IR among the anomaly free symmetries of the full theory. Since the dynamics is invariant under interchange of Q and $\tilde{Q}$, one must have $R(Q) = R(\tilde{Q})$. There is then a

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unique solution to the anomaly constraint[2] (1.5):

$$R(Q) = R(\tilde{Q}) = \frac{N_f - N_c}{N_f}\,. \tag{1.6}$$

The gauge invariant chiral operators

$$\begin{aligned} \mathcal{M}^i_{\tilde{i}} &= Q^i \tilde{Q}_{\tilde{i}}\,, \\ \mathcal{B}^{[i_1,\ldots,i_{N_c}]} &= Q^{i_1}\cdots Q^{i_{N_c}}\,, \\ \tilde{\mathcal{B}}_{[\tilde{i}_1,\ldots,\tilde{i}_{N_c}]} &= \tilde{Q}_{\tilde{i}_1}\cdots \tilde{Q}_{\tilde{i}_{N_c}}\,, \end{aligned} \tag{1.7}$$

are assigned R-charges

$$\begin{aligned} R(\mathcal{M}) &= 2\frac{(N_f - N_c)}{N_f}\,, \\ R(\mathcal{B}) &= R(\tilde{\mathcal{B}}) = \frac{N_c(N_f - N_c)}{N_f}\,. \end{aligned} \tag{1.8}$$

Eq. (1.8) is known to break down at $N_f = 3N_c/2$, illustrating both of the phenomena mentioned above. First, for $N_f < 3N_c/2$, $R(\mathcal{M}) < 2/3$, and as explained above the chiral superfield $\mathcal{M}$ (1.7) becomes free in the IR. This is an example of point (1) above.

One might expect that the prediction for $R(\mathcal{B})$ in (1.8) should still be valid, since it is typically large in the range of N_f, N_c under consideration, but this is known to be incorrect. In fact, the right description for $N_f < 3N_c/2$ is in terms of a Seiberg dual theory [15], with gauge group $\mathrm{SU}(N_f - N_c)$. The baryons $\mathcal{B}$ (1.7) can be expressed in terms of the magnetic quarks q as $\mathcal{B} \sim q^{N_f - N_c}$. Since the latter are free for $N_c < N_f \leq 3N_c/2$, one has

$$R(\mathcal{B}) = \frac{2(N_f - N_c)}{3}\,. \tag{1.9}$$

We see that (1.8) fails for $N_f < 3N_c/2$, even though $R(\mathcal{B})$ is typically large and positive when that happens. This is an example of point (2) above. It is important to emphasize that while in this case both of the kinds of violations discussed in points (1) and (2) above occur in the same regime, $N_f < 3N_c/2$, in general these two types of phenomena are distinct, and we will see examples later where only one or the other occurs.

To summarize, we see that there is a finite region, $3N_c/2 \leq N_f \leq 3N_c$ in which one can identify the IR R-charge as a symmetry of the full theory. For $N_f < 3N_c/2$ this idea fails, but then Seiberg duality comes to the rescue and allows one to solve the problem using a weakly coupled description.

In more general SYM theories the situation is expected to be qualitatively similar to that described above. Imagine, for simplicity, that there is a parameter, like N_f, as a function of which the IR coupling varies between zero and some finite (or infinite) value. Then, one expects to find a range of parameters for which the IR $\mathrm{U}(1)_R$ is one of the anomaly free symmetries (1.5), perhaps corrected by taking into account the decoupling of some free fields, as in point (1) above. Beyond this range, this method breaks down and one needs to proceed in some other way (e.g. use Seiberg duality).

[2]For $\mathrm{SU}(N_c)$, $T(\text{fund}) = 1/2$, $T(\text{adjoint}) = N_c$.

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The first step of this process, finding the solution of (1.5) that corresponds to the IR $U(1)_R$, is in general non-trivial since the solution is not unique. For example, in the case of interest in this paper, adjoint SQCD, i.e. SYM with gauge group $G = SU(N_c)$ and matter superfields X, Q^i, $\tilde{Q}_{\tilde{i}}$ in the adjoint, fundamental and anti-fundamental representations of G, respectively ($i, \tilde{i} = 1, \ldots, N_f$), assigning R-charge $R(Q)$ to Q, $\tilde{Q}$ and $R(X)$ to X, one has (1.5):

$$N_f R(Q) + N_c R(X) = N_f \,. \tag{1.10}$$

Thus, there is a one parameter set of candidate R-symmetries, and it is not clear which of these is the correct one. This is the problem solved by IW [1]. These authors proved that if the IR R-charge is a solution of (1.5), it is the one that locally maximizes a (1.1) over the set of all solutions of (1.5).[3]

From the point of view of the general discussion above, a few natural questions are:

(a) Does the R-symmetry found by IW describe the IR fixed point for all N_f, or does it fail for $N_f < N_f^*$, with some $N_f^* > 0$?

(b) What does the R-symmetry of IW imply for the possible flows corresponding to relevant deformations of the IR fixed point?

(c) Are the RG flows implied by the IW results consistent with the a-theorem?

The purpose of this paper is to discuss these issues. We will find that the results of [1] (corrected slightly to take into account unitarity constraints) lead to a sensible picture of the structure of RG flows in adjoint SQCD, which is in particular consistent with the results of [16, 17, 18] on Seiberg duality in these models, and with the a-theorem. The potential violations of the a-theorem pointed out in [8], as well as others, are avoided by these flows. In the theory with vanishing superpotential for X, Q^i, $\tilde{Q}_{\tilde{i}}$ we do not find any evidence for the breakdown of the results of [1] for any $N_f > 0$, while in the theories with a non-zero superpotential we exhibit examples in which such an analysis has a limited domain of validity, and in order to explore the whole phase diagram one has to appeal to Seiberg duality.

The plan of the paper is as follows. In section 2 we study adjoint SQCD with vanishing superpotential for large N_f, N_c, with fixed N_c/N_f. We apply the analysis of IW to this case, taking into account unitarity constraints which modify slightly the results given in [1], compute the R-charges of Q and X, and the resulting central charge $\tilde{a}$. We show that the a-theorem is satisfied for the RG flows in this system.

In section 3 we study deformations of adjoint SQCD corresponding to Higgsing the gauge group and turning on a polynomial superpotential for the adjoint superfield X. Again, the a-theorem appears to be satisfied, rather non-trivially.

In section 4 we discuss the dynamics of the system in the presence of a polynomial superpotential, by using a dual description due to [16]–[18]. We show that the results of section 3 lead to a picture consistent with the duality. At the same time, the duality

[3] As we will see below, the situation is actually slightly more involved.

predicts that the calculation of the central charge $\tilde{a}$ done in section 3 breaks down at some critical value of N_c/N_f, which we find. Beyond that point one must switch to the dual variables in order to compute it correctly.

Section 5 contains a brief discussion. In appendix A we derive some technical results that are used in the text.

2. Adjoint SQCD with vanishing superpotential

In this section we will study adjoint SQCD, which was mentioned in the introduction, using the results of [1]. For simplicity, we will work in the large-N limit[4]

$$N_c \gg 1\,; \qquad N_f \gg 1\,; \qquad x \equiv \frac{N_c}{N_f} = \text{fixed}\,, \tag{2.1}$$

and study the phase structure as a function of the continuous parameter x. The theory is asymptotically free for $x > 1/2$, and we will mostly restrict our discussion to this regime.

Under RG flow, the gauge theory in question flows in the IR to a non-trivial fixed point. To find the $\mathrm{U}(1)_R$ symmetry at that fixed point we follow [1]. We assign R-charge y to Q, $\tilde{Q}$, and compute the "trial" central charge (1.1), which we will denote by $\tilde{a}^{(0)}(x,y)$, using (1.10) to express the R-charge of X as a function of y, $R(X) = (1-y)/x$. One finds:

$$\tilde{a}^{(0)}(x,y)/N_f^2 = 6x(y-1)^3 - 2x(y-1) + 3x^2\left(\frac{1-y}{x} - 1\right)^3 - x^2\left(\frac{1-y}{x} - 1\right) + 2x^2\,. \tag{2.2}$$

As shown in [1], the IR $\mathrm{U}(1)_R$ can be determined by requiring that the trial central charge $\tilde{a}^{(0)}(x,y)$ (2.2) is at a local maximum with respect to y. This leads to:

$$\begin{aligned} R^{(0)}(Q) &= y^{(0)} = \frac{3 + x(-3 - 6x + \sqrt{20x^2 - 1})}{3 - 6x^2}\,, \\ R^{(0)}(X) &= \frac{1 - y^{(0)}}{x} = \frac{10}{3(3 + \sqrt{20x^2 - 1})}\,. \end{aligned} \tag{2.3}$$

Plugging (2.3) into (2.2), one finds the following expression for the central charge:

$$\tilde{a}^{(0)}(x) = \frac{2N_f^2 x^2}{9(1 - 2x^2)^2}\left[18 - 90x^2 + (20x^2 - 1)^{3/2}\right]. \tag{2.4}$$

Actually, (2.3), (2.4) are not the full story, since one needs to take into account unitarity constraints. Consider, for example, the chiral superfield $\mathcal{M}_1 = \tilde{Q}Q$. According to (2.3), it violates the unitarity bound for $x > 3 + \sqrt{7}$, and the procedure of [1] must be modified in this regime.

As discussed in the introduction, it is believed that in this situation the infrared SCFT splits into an (in general) interacting theory and a decoupled free superfield $\mathcal{M}_1$. The trial central charge $\tilde{a}^{(0)}$ (2.2) can then be written as a sum of two contributions. One comes from the decoupled superfield $\mathcal{M}_1$,

$$\tilde{a}_M(R(\mathcal{M}_1))/N_f^2 = 3\left[R(\mathcal{M}_1) - 1\right]^3 - \left[R(\mathcal{M}_1) - 1\right], \tag{2.5}$$

[4] It is not difficult to extend the analysis to finite N_c, N_f.

where $R(\mathcal{M}_1) = 2R(Q) = 2y$; the other contribution is due to the interacting SCFT,

$$\tilde{a}_{\text{interact}} = \tilde{a}^{(0)}(x,y) - \tilde{a}_M(R(\mathcal{M}_1))\,. \tag{2.6}$$

It is clear that in order to find the IR $\mathrm{U}(1)_R$ we only have to extremize $\tilde{a}_{\text{interact}}$, since we know what to do with the free superfield $\mathcal{M}_1$. Thus, for $x > 3+\sqrt{7}$, the results (2.3) are invalid, and are replaced by those following from the extremization of (2.6). The full central charge is [8]

$$\tilde{a}^{(1)} = \tilde{a}_{\text{interact}} + \tilde{a}_M\left(\frac{2}{3}\right). \tag{2.7}$$

This is not the end of the story either, since at some yet larger value of x, the operator $\mathcal{M}_2 = \tilde{Q}XQ$ reaches R-charge 2/3, and the same procedure has to be repeated for it. More generally, every time the R-charge of a gauge invariant chiral operator in the theory drops below 2/3, the procedure of [1] has to be modified accordingly.

In practice one can proceed as follows. Denote by M the collection of gauge invariant chiral superfields whose R-charge is smaller than 2/3 for a particular value of x (or N_f). The trial central charge that one should extremize is

$$\begin{aligned}\tilde{a} &= \tilde{a}^{(0)}(x,y) + \sum_M \left[\tilde{a}_M\left(\frac{2}{3}\right) - \tilde{a}_M(R(M))\right] \\ &= \tilde{a}^{(0)}(x,y) + \frac{1}{9}\sum_M \dim(M)\,[2-3R(M)]^2\,[5-3R(M)]\,,\end{aligned} \tag{2.8}$$

where $R(M)$ is the R-charge of M under the trial R-symmetry, $\dim(M)$ is the number of fields with the same R-charge, and the sum over M runs over fields with different R-charges. It is easy to generalize the above discussion to other gauge groups and matter contents.

In adjoint SQCD there are three types of gauge invariant chiral superfields that are relevant for the preceding discussion:

$$\begin{aligned}&\mathcal{B}^{(n_1,n_2,\dots,n_k)} = Q_{(1)}^{n_1}\cdots Q_{(k)}^{n_k}\,; \qquad \sum_{l=1}^{k} n_l = N_c\,; \quad n_l \le N_f\,; \quad k=1,2,\dots\,,\\ &\mathrm{tr}\,X^j\,, \qquad j=2,3,\dots\,,\\ &(\mathcal{M}_j)_{\tilde{i}}^{i} = \tilde{Q}_{\tilde{i}}X^{j-1}Q^i\,, \qquad j=1,2,\dots\,,\end{aligned} \tag{2.9}$$

where in the first line the color indices are contracted with an ϵ tensor, and following [17] we introduced "dressed quarks"

$$Q_{(l)} = X^{l-1}Q\,; \qquad l=1,2,\dots\,. \tag{2.10}$$

There are also baryons $\tilde{\mathcal{B}}$ obtained from (2.9) by replacing $Q\to\tilde{Q}$. The R-charge of the baryons $\mathcal{B}$ is given by

$$R(\mathcal{B}^{(n_1,n_2,\dots,n_k)}) = \sum_{l=1}^{k} n_l(l-1)R(X) + N_c R(Q)\,. \tag{2.11}$$

In the limit we are considering (2.1), the R-charge of baryons is positive and infinite since, as we will see later, $R(X)$ and $R(Q)$ are both positive. Therefore, $\mathcal{B}$ and $\tilde{\mathcal{B}}$ do not contribute to the correction terms in (2.8). The fields on the second line of (2.9) do not contribute to these corrections either, even if their R-charge reaches 2/3, since in the limit (2.1) their contribution to $\tilde{a}$ is smaller than the other terms by a factor of N_f^2. On the other hand the fields $(\mathcal{M}_j)_{\tilde{i}}^{i}$ can potentially change $\tilde{a}$ in the limit that we are considering, since there are N_f^2 of them for each j. The R-charge of $(\mathcal{M}_j)_{\tilde{i}}^{i}$ computed as a composite field is given by

$$R(\mathcal{M}_j) = 2y + (j-1)\frac{1-y}{x}\,. \tag{2.12}$$

We will see later that for large x, y approaches a constant $y_0 < 1/3$. Thus, for any given j there exists a value of x above which $\mathcal{M}_j$ becomes free.

In order to determine $\tilde{a}$, it is convenient to introduce an auxiliary quantity, the trial central charge computed with the assumption that the first $p(=0,1,2,\ldots)$ meson fields $\mathcal{M}_1,\ldots,\mathcal{M}_p$ are free:

$$\begin{aligned}\frac{\tilde{a}^{(p)}(x,y)}{N_f^2} &= 6x(y-1)^3 - 2x(y-1) + 3x^2\left(\frac{1-y}{x}-1\right)^3 - x^2\left(\frac{1-y}{x}-1\right) + 2x^2 + \\ &\quad + \frac{1}{9}\sum_{j=1}^{p}\left[2-3\left(2y+(j-1)\frac{1-y}{x}\right)\right]^2\left[5-3\left(2y+(j-1)\frac{1-y}{x}\right)\right]. \end{aligned} \tag{2.13}$$

The central charge $\tilde{a}$ can then be determined via the following process:

1. Maximize each $\tilde{a}^{(p)}$ with respect to y, and find the corresponding value of y, $y^{(p)}(x)$.
2. Substitute $y^{(p)}$ into (2.12) and compute $\tilde{p}(p,x)$, such that $\mathcal{M}_1\cdots\mathcal{M}_{\tilde{p}}$ have $R \le 2/3$, but $R(\mathcal{M}_{\tilde{p}+1}) > 2/3$.
3. There is a unique value of p, p_0, for which[5] $\tilde{p}(p_0,x) = p_0$. The central charge $\tilde{a}(x)$ is given by $\tilde{a}(x) = \tilde{a}^{(p_0)}(x, y^{(p_0)}(x))$.

It is useful to note that according to (2.13), the p-th meson $\mathcal{M}_p$ becomes free when

$$\begin{aligned}\tilde{a}^{(p-1)} &= \tilde{a}^{(p)}\,,\\ \partial_y\tilde{a}^{(p-1)} &= \partial_y\tilde{a}^{(p)}\,.\end{aligned} \tag{2.14}$$

Therefore, the R-charges computed from $\tilde{a}^{(p-1)}$, $\tilde{a}^{(p)}$ coincide at this point. Moreover, at this point

$$\begin{aligned}\tilde{a}^{(p-1)}(x,y^{(p-1)}(x)) &= \tilde{a}^{(p)}(x,y^{(p)}(x))\,,\\ \frac{d}{dx}\tilde{a}^{(p-1)}(x,y^{(p-1)}(x)) &= \frac{d}{dx}\tilde{a}^{(p)}(x,y^{(p)}(x))\,,\end{aligned} \tag{2.15}$$

which means that the central charge $\tilde{a}$ is a continuous, smooth function of x. One can set up an iterative algorithm for finding the central charge $\tilde{a}(x)$ and the R-charges $R(Q)$

[5]Note that p_0 depends on x.

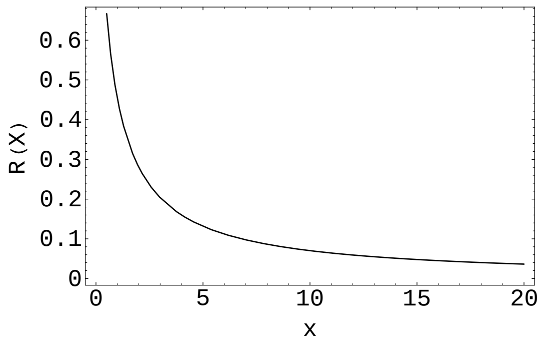

Figure 1: The R-charge of X as a function of x.

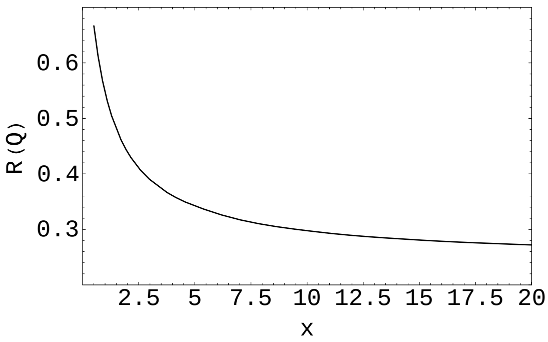

Figure 2: The R-charge of Q as a function of x.

and $R(X)$ as follows. Start with $\tilde{a}^{(0)}$ and find the value of x for which the R-charge of $\mathcal{M}_1 = \tilde{Q}Q$ approaches $2/3$. At that point $\mathcal{M}_1$ becomes free and we have to switch to the $\tilde{a}^{(1)}$ description. Then look for the value of x at which $\mathcal{M}_2$ becomes free and decouples, switch to $\tilde{a}^{(2)}$, etc. This process can be obviously continued to arbitrarily large x.

We have implemented this algorithm using Mathematica. The results for the R-charges are given in figures 1, 2. A few comments are in order regarding this procedure:

1. It assumes that the mesons become free sequentially, i.e. at the point when $\mathcal{M}_p$ becomes free, all the mesons $\mathcal{M}_j$ with $j < p$ are already free. This is indeed the case provided $R(X) > 0$, as one can see from (2.12).

2. It is a logical possibility that the R-charge of the p'th meson may cross the line $R = 2/3$ more then once. Then one would have to switch back to the description in terms of $\tilde{a}^{(p-1)}$ when this happens the second time. The results presented in figures 1, 2 indicate that this possibility is not realized, since $R(Q)$ and $R(X)$ are positive and monotonically decreasing functions of x.

3. The R-charge $R(Q)$ (and consequently $R(X)$) obtained using the algorithm described above is different from that obtained from (2.3), $R^{(0)}(Q)$. For example, the asymptotic value of $R(Q)$ as $x \to \infty$ is (see appendix A)

$$y_{as} = \frac{\sqrt{3}-1}{3} \simeq 0.24402\,, \tag{2.16}$$

while (2.3) leads to

$$y'_{as} = \frac{3-\sqrt{5}}{3} \simeq 0.25464\,. \tag{2.17}$$

In figure 3 we plot the difference between $R_0(Q)$ and $R(Q)$. Comparing figures 2 and 3, we see that the quantitative difference between the two is rather small (at most at the few percent level).

To conclude this section we would like to discuss the implications of our analysis for the a-theorem. The latter predicts that for $x > 1/2$,

$$\tilde{a}_{UV} > \tilde{a}_{IR} > 0\,. \tag{2.18}$$

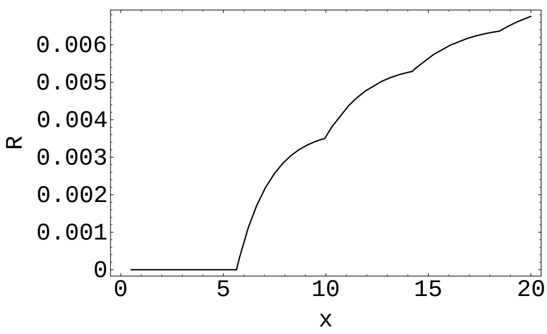

Figure 3: The difference $R_0(Q) - R(Q)$ as a function of x. This function is not smooth at the points where the mesons $\mathcal{M}_p$ ($p = 1, 2, \ldots$) decouple.

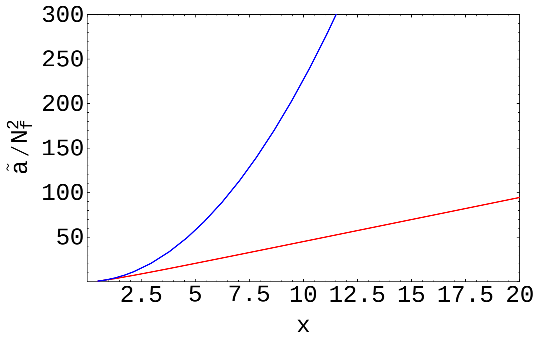

Figure 4: The UV (top, blue curve) and IR (bottom, red curve) values of the central charge $\tilde{a}/N_f^2$ as a function of x.

Since the UV theory is free, one has

$$\tilde{a}_{UV} = \frac{2}{9} N_c N_f \cdot 2 + \frac{2}{9} N_c^2 + 2N_c^2 = N_f^2 \left[\frac{4}{9} x + \frac{20}{9} x^2 \right], \tag{2.19}$$

where we listed the contribution of the fundamentals Q, $\tilde{Q}$, followed by that of the adjoint field X, and of the gauginos.

In figure 4 we plot $\tilde{a}_{UV}$ and $\tilde{a}_{IR}$. We see that (2.18) is indeed satisfied. Figure 4 also shows that, as seen in the asymptotic large x analysis of appendix A, $\tilde{a}_{IR}$ does not contain a term quadratic in x as $x \to \infty$, in contrast to $\tilde{a}_{UV}$ (2.19). By fitting the data leading to figure 4 to an asymptotically linear function, one finds

$$\frac{\tilde{a}_{IR}}{N_f^2} = cx + d + O\left(\frac{1}{x}\right), \tag{2.20}$$

where

$$c \simeq 4.976\,; \qquad d \simeq -5.08\,. \tag{2.21}$$

The value of c agrees with the analytic result obtained in appendix A, $c = 4(2+\sqrt{3})/3$. The fact that d is negative will be seen later to be a necessary condition for the validity of the a-theorem.

A slightly more sensitive test of the positivity of $\tilde{a}$ than figure 4 is to compute only the contribution of the interacting part of the IR CFT, which can be obtained by a generalization of (2.6)

$$\tilde{a}_{\text{interact}}(x) = \tilde{a}(x) - p\tilde{a}_M\left(\frac{2}{3}\right) = \tilde{a}^{(0)}(x, y(x)) - \sum_{j=1}^{p} \tilde{a}_M(R(\mathcal{M}_j))\,, \tag{2.22}$$

where p is the number of mesons which are free at x. We plot the difference of the full central charge and this quantity in figure 5. Comparing figures 4 and 5 we see that the contribution of the interacting part of the CFT is always positive, and in fact is always much larger than the contribution of the decoupled free fields.

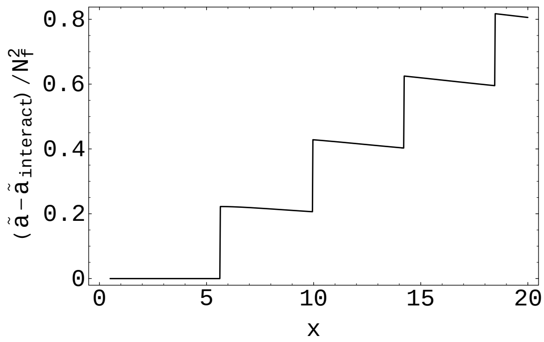

Figure 5: The difference between the full central charge, $\tilde{a}$, and the contribution of the interacting part of the IR SCFT, $\tilde{a}_{\text{interact}}$, as a function of x. Discontinuities correspond to points where additional mesons become free.

3. Deformations of adjoint SQCD

In this section we use the improved understanding of the infrared behavior of adjoint SQCD discussed in section 2, to study perturbations of that fixed point. There are two different types of deformations that one can consider: giving v.e.v.'s to massless scalars with vanishing potentials, and perturbing the lagrangian by relevant operators. We will next consider these two types of perturbations in turn.

3.1 Higgsing

Adjoint SQCD has a classical moduli space whose (complex) dimension is $2N_fN_c$. It is believed that the quantum theory has a moduli space of the same dimension. Here we will discuss a particular subspace of the moduli space, corresponding to turning on expectation values of the adjoint superfield X,

$$\langle X\rangle = \mathrm{diag}(\alpha_1^{n_1}, \alpha_2^{n_2}, \dots, \alpha_l^{n_l})\,; \qquad \sum_j n_j = N_c\,, \tag{3.1}$$

i.e. the first n_1 eigenvalues of X are equal to α_1, the next n_2 are equal to α_2, etc. By definition, α_j, $j = 1, 2, \dots, l$ are all distinct. The v.e.v. (3.1) corresponds to a relevant deformation of the theory; it leads in the infrared to a direct product of $\mathrm{SU}(n_j)$ adjoint SQCD theories,[6] each of which flows to a fixed point of the sort discussed in section 2. The corresponding central charge $\tilde{a}$ is the sum of the central charges of the individual factors. Defining the variables $x_j = n_j/N_f$, in analogy to (2.1), the a-theorem implies that

$$\tilde{a}(x) > \sum_{j=1}^{l} \tilde{a}(x_j)\,, \tag{3.2}$$

where $x_j > 0$ and

$$\sum_{j=1}^{l} x_j = x\,. \tag{3.3}$$

Equation (3.2) is a non-trivial requirement on the function $\tilde{a}$. It is not clear to us how to prove or disprove it in general; we next check its validity in a few cases.

[6]There are also some U(1) factors in the gauge group, but these can be ignored in the large-N limit (2.1).

Consider, for example, the symmetry breaking pattern $\mathrm{SU}(N_c) \to [\mathrm{SU}(N_c/l)]^l$ corresponding to $n_1 = n_2 = \cdots = n_l = N_c/l$. In this case all the x_j are equal to each other, and by (3.3), $x_j = x/l \equiv b$. The condition (3.2) is now

$$\tilde{a}(x) > \frac{x}{b}\tilde{a}(b)\,. \tag{3.4}$$

Since x can be taken to be arbitrarily large, comparing to (2.20) we see that the a-theorem leads to the constraint

$$\frac{\tilde{a}(b)}{bN_f^2} < c = \frac{4(2+\sqrt{3})}{3}\,, \tag{3.5}$$

for all b. The simplest case is $b < 1/2$. In this case, the $\mathrm{SU}(N_c/l)$ theories one finds in the infrared are not asymptotically free, and their central charge is given by the free field theory result, (2.19):

$$\frac{\tilde{a}(b)}{N_f^2} = \frac{4}{9}b + \frac{20}{9}b^2\,. \tag{3.6}$$

The constraint (3.5) now reads

$$\frac{4}{9} + \frac{20}{9}b < \frac{4(2+\sqrt{3})}{3}\,. \tag{3.7}$$

It is indeed valid for $b < 1/2$, the region of validity of (3.6). It would have been violated when $b \simeq 2$, but in that regime one has to use (2.4) for $\tilde{a}(b)$, and one can check that it too satisfies the constraint (3.5). More generally, one can check that (3.5) is satisfied for all b (see figure 6).

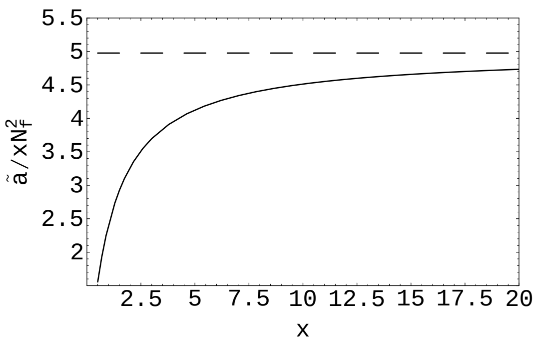

Figure 6: $\tilde{a}(x)/xN_f^2$ (solid line) is bounded from above by $c = 4(2+\sqrt{3})/3$ (dashed line), and approaches it for large x. This agrees with the expectation from the a-theorem, (3.5).

Another simple check of (3.2) is obtained by considering the case where both x and the x_j are very large. In that case one can use the asymptotic formula (2.20) both for the left and right hand sides of (3.2):

$$cx + d > \sum_{j=1}^{l}(cx_j + d)\,. \tag{3.8}$$

The leading term (linear in x) is the same on both sides (see (3.3)). The inequality is indeed satisfied since d is negative (see (2.21)). A positive d would lead to a violation of the a-theorem; if d vanished, one would have had to consider the $O(1/x)$ term in (2.20).

3.2 Relevant superpotential perturbations

Another interesting class of deformations of adjoint SQCD corresponds to relevant perturbations of the superpotential (F terms). These perturbations can be expressed in terms of the chiral operators in eq. (2.9). The baryons (first line of (2.9)) are in general irrelevant

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and can be ignored. We will focus here on perturbations by the operators on the second line of (2.9),

$$W_k(X) = g_k \,\mathrm{tr}\, X^{k+1} . \tag{3.9}$$

It would be interesting to generalize the discussion to perturbations by the mesons on the third line of (2.9), $\mathcal{M}_j$.

As discussed in the introduction, although the deformations (3.9) are irrelevant in the UV for all $k > 1$ (the case $k = 1$ corresponds to a mass term for X), they might lead to relevant deformations of the IR fixed point of adjoint SQCD. Indeed, we saw in section 2 that as we vary x (2.1), the R-charge of X monotonically decreases, approaching zero at large x as

$$R(X) \simeq \frac{4-\sqrt{3}}{3x} . \tag{3.10}$$

Therefore, for any given k there always exists an x_k such that

$$R(X^{k+1}) \leq 2 , \qquad \text{for } x \geq x_k . \tag{3.11}$$

For $x > x_k$, $\mathrm{tr}\, X^{k+1}$ is a relevant deformation of the infrared fixed point and by turning it on we expect to flow to a new fixed point. Assuming that the R-charge at this fixed point should be visible throughout the RG flow from the UV, one can determine it using (1.10) and the condition that W_k is marginal [16, 17],

$$y_k = R_k(Q) = R_k(\tilde{Q}) = 1 - \frac{2}{k+1} x , \qquad R_k(X) = \frac{2}{k+1} . \tag{3.12}$$

It was shown in [17] that the theory with the perturbation (3.9) turned on has a stable vacuum only for $N_f \geq N_c/k$, or

$$x \leq k . \tag{3.13}$$

This was done by deforming the superpotential (3.9) to a generic polynomial of degree $k+1$ in X, using the results of [19, 20] to show that the resulting model has no vacuum for $x > k$, and then removing the perturbation and going back to (3.9).

Implicit in the argument of [17] was the assertion that the perturbation (3.9) must become relevant before one reaches the point $x = k$, where it destabilizes the theory. Indeed, it would be inconsistent for a perturbation which is irrelevant at the IR fixed point of the gauge theory to destabilize that fixed point at long distances. Therefore, we conclude that it must be that (see (3.11))

$$x_k < k . \tag{3.14}$$

At the time [17] was written, x_k was not known, but now, using the results of [1] and section 2 of this paper, we can verify (3.14). For small k, one can use (2.3) and (3.11), which lead to

$$x_k = \sqrt{\frac{1}{20}\left(\frac{(5k-4)^2}{9} + 1\right)} . \tag{3.15}$$

For example, for $k = 2$, which corresponds to a cubic superpotential (3.9), one finds $x_2 = 1/2$. Indeed, this perturbation is marginally irrelevant at the UV fixed point of the gauge theory, but is relevant at the IR fixed point whenever the gauge theory is asymptotically free.

Recall that (2.3) is valid only for $x \leq 3+\sqrt{7}$. This implies that (3.15) is only valid for $k \leq 15$. For larger k, the full analysis of section 2 is needed. For $k \gg 1$, using the asymptotic form of the solution, (3.10), one finds

$$x_k \simeq \frac{4-\sqrt{3}}{6} k \,. \tag{3.16}$$

It is easy to verify that both (3.15) and (3.16) satisfy the constraint (3.14). In fact, since the quantitative difference between the IW result (3.15) and the exact result is small, one can use (3.15) to verify (3.14) for all k.

One can actually provide a better bound on x_k as follows. The fact that the R-charges $R(Q)=y$ and $R(X)=(1-y)/x$ are monotonically decreasing functions of x can be used to prove the following inequality

$$\frac{x_{k+1}}{k+2} > \frac{x_k}{k+1} \,. \tag{3.17}$$

Indeed

$$\frac{x_{k+1}}{k+2} = \frac{1-y(x_{k+1})}{2} > \frac{1-y(x_k)}{2} = \frac{x_k}{k+1} \,, \tag{3.18}$$

where we used the fact that y is a monotonically decreasing function of x (see figure 2). Furthermore, (3.17), (3.18) imply that

$$\frac{x_k}{k+1} < \frac{1-y_{as}}{2} = \frac{1}{2}\left(1-\frac{\sqrt{3}-1}{3}\right) , \tag{3.19}$$

which is a stronger bound than (3.14) and will be useful for other purposes below. Note that the bound (3.19) is saturated in the limit $k \to \infty$ (see (3.16)).

We would next like to apply the results discussed above to the different flows associated with the superpotentials (3.9). We will focus on two types of flows. The first is the flow from the IR fixed point of adjoint SQCD with $W=0$ to the fixed point $\mathbf{k}$ obtained by turning on g_k (3.9). The second is the flow from $\mathbf{k}$ to $\mathbf{k}'$, with $k'<k$. This flow is obtained by studying a superpotential of the form

$$W(X) = g_k \operatorname{tr} X^{k+1} + g_{k'} \operatorname{tr} X^{k'+1} \,. \tag{3.20}$$

We first set $g_{k'}=0$, flow to the infrared fixed point $\mathbf{k}$, and then turn on $g_{k'}$ to further flow to $\mathbf{k}'$. As discussed above, g_k corresponds to a relevant perturbation only for $x>x_k$, so we should restrict consideration to this range (and take into account the condition for having a stable vacuum (3.13), $x \leq k$).

One of the main questions we would like to address is the validity of the a-theorem along these flows. To compute the central charge a, we must determine the gauge invariant chiral operators which might become free as we vary x. The chiral ring of the gauge theory with the superpotential (3.9) is generated by the operators [16]–[18]

$$\begin{aligned}
&\mathcal{B}^{(n_1,n_2,\ldots,n_k)} = Q_{(1)}^{n_1}\cdots Q_{(k)}^{n_k}\,; \qquad \sum_{l=1}^{k} n_i = N_c\,; \quad n_i \leq N_f\,,\\
&\operatorname{tr} X^2, \operatorname{tr} X^3, \ldots, \operatorname{tr} X^k\,,\\
&(\mathcal{M}_j)_i^{\tilde{i}} = Q_i X^{j-1} Q^{\tilde{i}}\,, \qquad j=1,\ldots,k\,.
\end{aligned} \tag{3.21}$$

$Q_{(l)}$ are defined by (2.10). As in section 2, the operators $\operatorname{tr} X^l$ do not contribute to the corrections to $\tilde{a}$ in the limit (2.1) because their contribution is down by a factor of N_f^2. The baryons $\mathcal{B}$ need to be examined more carefully than in section 2, since as one can see from (3.12), $R_k(Q)$ becomes negative for

$$x > \frac{k+1}{2}\,, \tag{3.22}$$

so the R-charge of the baryons, which is given as before by

$$R(\mathcal{B}^{(n_1,n_2,\ldots,n_k)}) = \sum_{i=1}^{k} n_i(i-1)R_k(X) + N_c R_k(Q)\,, \tag{3.23}$$

can potentially violate the unitarity bound. We show below that for $x \leq k$ the R-charge of the baryons is positive (and infinite in the limit (2.1)) so there should not be any corrections of type (2.8) associated with them. Indeed, plugging (3.12) in (3.23) one finds

$$R(\mathcal{B}^{(n_1,n_2,\ldots,n_k)}) = \frac{2}{k+1}\left[N_c\left(\frac{k-1}{2} - x\right) + \sum_{i=1}^{k} n_i i\right]. \tag{3.24}$$

To find a lower bound on $R(\mathcal{B}^{(n_1,n_2,\ldots,n_k)})$, we would like to find a lower bound on $\sum_{i=1}^{k} n_i i$. It is easy to see that

$$\sum_{i=1}^{k} n_i i \geq N_f \sum_{i=1}^{[x]} i + (N_c - [x]N_f)([x]+1)i = N_f\frac{[x]([x]+1)}{2} + (N_c - [x]N_f)([x]+1)\,, \tag{3.25}$$

where $[x]$ is the integer part of x. Indeed, $\sum_{i=1}^{k} n_i i$ is minimized by the following choice of the n_i

$$n_1 = \cdots = n_{[x]} = N_f\,; \qquad n_{[x]+1} = N_c - [x]N_f\,; \qquad n_i = 0\,, \quad \text{for } i = [x]+2, \ldots, k\,. \tag{3.26}$$

Eq. (3.25) leads to the following bound on the R-charge of the baryons:

$$R(\mathcal{B}^{(n_1,n_2,\ldots,n_k)}) \geq \frac{N_f}{k+1}\left\{x(k+1) - [x] - (2x^2 + [x]^2 - 2x[x])\right\}. \tag{3.27}$$

The expression on the right hand side of (3.27) is a monotonically decreasing function of x for $x > (k+1)/2$. It vanishes[7] at $x = k$. We conclude that the R-charges of baryons are large and positive, and thus baryons do not contribute to the correction terms in $\tilde{a}_k$. The mesons $\mathcal{M}_j$ do contribute to these correction terms, as in the discussion of section 2. Their R-charges are given by

$$R(\mathcal{M}_j) = 2y_k + (j-1)\frac{1-y_k}{x} = 2\frac{j+k-2x}{k+1}\,. \tag{3.28}$$

[7]For $x > k$, even separately from the fact that the quantum theory has no vacuum, the baryons (3.21) do not exist, so the issue of their R-charge does not arise.

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Plugging into (2.8) one finds

$$\begin{aligned}\frac{\tilde{a}_k(x)}{N_f^2} &= 6x(y_k-1)^3 - 2x(y_k-1) + 3x^2\left(\frac{1-y_k}{x}-1\right)^3 - x^2\left(\frac{1-y_k}{x}-1\right) + 2x^2 + \\ &\quad + \frac{1}{9}\sum_{j=1}^{p(x)}\left[2-3\left(2y_k+(j-1)\frac{1-y_k}{x}\right)\right]^2\left[5-3\left(2y_k+(j-1)\frac{1-y_k}{x}\right)\right] \\ &= \frac{4}{(k+1)^3}\left[x^2(2+k+5k^2-12x^2)-\frac{1}{9}\sum_{j=1}^{p(x)}(-5+6j+k-12x)(1-3j-2k+6x)^2\right].\end{aligned} \tag{3.29}$$

Here $p(x)$ is the number of mesons which are free at x,

$$p(x) = \begin{cases} \left[\frac{1}{3}(6x-2k+1)\right] & \text{if } \left[\frac{1}{3}(6x-2k+1)\right] \leq k \\ k & \text{otherwise,} \end{cases} \tag{3.30}$$

where $[\ldots]$ is the integer part of the expression in brackets if the expression is positive and 0 otherwise. The p-th meson becomes free at

$$x(p) = \frac{1}{6}(3p+2k-1)\,. \tag{3.31}$$

Using eq. (3.16) as an estimate for x_k, we see that at $x \simeq x_k$, some mesons have already decoupled (the precise number depends on k).

We are now ready to consider the RG flows associated with the polynomial superpotentials (3.9), (3.20). Consider first the flow from the infrared fixed point of adjoint SQCD to the fixed point $\mathbf{k}$ obtained by switching on (3.9). The a-theorem expectation is

$$\tilde{a}(x) > \tilde{a}_k(x) > 0\,, \qquad \text{for } k > x > x_k\,. \tag{3.32}$$

As pointed out in [1], (3.32) is guaranteed to be satisfied in a finite region near $x = x_k$, by construction. Indeed, $\tilde{a}(x)$ was found in section 2 by maximizing $a^{(p)}(x,y)$ (2.13) with respect to y. $\tilde{a}_k(x)$ is obtained by fixing y to a particular value (3.12). The two coincide at $x = x_k$. Therefore, if x is sufficiently close to x_k, (3.32) is satisfied. If x deviates significantly from x_k, this argument no longer applies, since the maximum leading to $\tilde{a}(x)$ is a *local* one.

We have not found a proof of (3.32), but our numerical results suggest that it is always satisfied. In figure 7 we exhibit the typical behavior of the central charges corresponding to the free UV fixed point, $\tilde{a}_{UV}(x)$, the IR fixed point of adjoint SQCD with $W = 0$, $\tilde{a}(x)$, and the fixed point $\mathbf{k}$ associated with (3.9), $\tilde{a}_k(x)$, for $k = 20$. The UV and IR curves for the $W = 0$ problem are as in section 2 (see figure 4). The point at which the two lower curves meet is $x = x_k$. As we saw earlier (eq. (3.19)), $1/2 < x_k < (k+1)/2$. The difference $\tilde{a}(x) - \tilde{a}_k(x)$ is plotted in figure 8.

Note that in figures 7, 8 we have extended the curve $\tilde{a}_k(x)$ to the full range $1/2 < x < k$. This is of course unphysical, since as discussed earlier, the fixed point $\mathbf{k}$ does not exist for $x < x_k$. Thus, the lower curve in figure 7 is unphysical to the left of the point where it

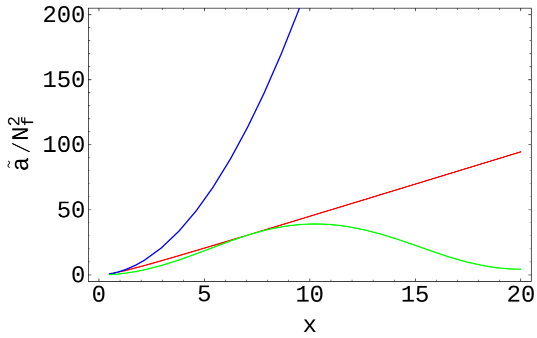

Figure 7: The central charge corresponding to the UV (top, blue curve) and IR (middle, red curve) fixed points of the theory with vanishing superpotential, and that of the fixed point $\mathbf{k}$ with $k = 20$ (bottom, green curve), as a function of x.

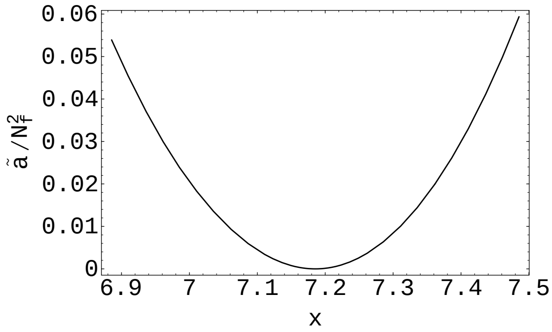

Figure 8: The difference $(\tilde{a}(x) - \tilde{a}_k(x))/N_f^2$ as a function of x; $k = 20$.

touches the curve above it. Similarly, the part of the curve of figure 8 to the left of the point where it touches the x axis is unphysical. At any rate, the main conclusion from figures 7, 8 is that the a-theorem seems to be satisfied for these flows.

It is worth noting that the corrections (2.8) are crucial for the positivity of $\tilde{a}_k$. In figure 9 we plot the uncorrected central charge, $\tilde{a}_k^{(0)}$, together with the corrected one, $\tilde{a}_k$. The former is actually negative in part of the physical domain $x_k < x < k$, while the latter is positive there.

We next turn to the flows $\mathbf{k} \to \mathbf{k}'$ associated with superpotentials of the form (3.20). The a-theorem predicts that

$$\tilde{a}_k > \tilde{a}_{k'} \,, \qquad \text{for } x > x_k \,. \tag{3.33}$$

The authors of [8] pointed out that in general, this prediction seems to be violated in a range of x's:

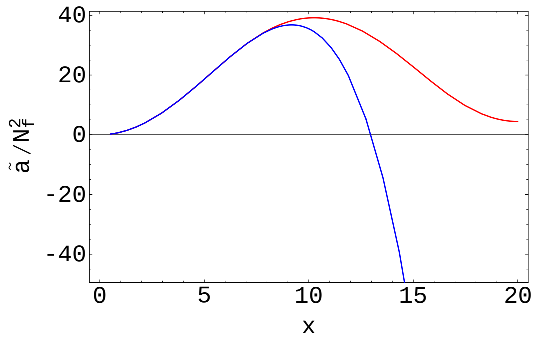

Figure 9: The central charge corresponding to the fixed point **20** associated with the superpotential $\mathrm{tr}X^{21}$. The top (red) curve takes into account the corrections due to the decoupling of meson fields discussed in the text, while the bottom (blue) curve neglects these corrections.

$$\tilde{a}_k < \tilde{a}_{k'} \,, \qquad \text{for } \frac{1}{2} < x < x_{\mathrm{int}}(k, k') \,. \tag{3.34}$$

They computed the value of $x_{\mathrm{int}}(k, k')$ in a few examples, obtaining the results

$$\begin{aligned} x_{\mathrm{int}}(3,2) &\simeq 0.65 \,, \\ x_{\mathrm{int}}(4,3) &\simeq 1.01 \,, \\ x_{\mathrm{int}}(5,4) &\simeq 1.38 \,. \end{aligned} \tag{3.35}$$

Thus, the a-theorem can potentially be violated in these flows, depending on the value of x_k. Comparing equations (3.33) and (3.34), we see that a necessary condition for the validity of the a-theorem is

$$x_k > x_{\rm int}(k,k')\,, \tag{3.36}$$

for all k, $k' < k$. At the time [8] was written, x_k was not known and thus it was impossible to check (3.36). Now, we can use the results of [1] and this paper to do that. For the cases (3.35), which involve small values of k, one can use (3.15):

$$\begin{aligned} x_3 &\simeq 0.85\,, \\ x_4 &\simeq 1.21\,, \\ x_5 &\simeq 1.58\,. \end{aligned} \tag{3.37}$$

Comparing to (3.35) we see that the necessary condition (3.36) is satisfied in all three cases.

Our numerical results seem to suggest that (3.36) is always satisfied. As an illustration, in figure 10 we plot the behavior of the various central charges for the case $k = 8$, $k' = 3$. The top curve is the central charge of the IR fixed point of the theory with vanishing superpotential. It intersects the two curves corresponding to the fixed points associated with the superpotentials trX^4 and trX^9 at the points $x = x_3$ and $x = x_8$, respectively (recall that $x_8 > x_3$; this can be used to determine which curve is which in figure 10). We see that a_8 is larger than a_3, in agreement with the a-theorem, for all $x > x_8$. The point at which the two curves cross, $x_{\rm int}(8,3)$ lies between x_3 and x_8 and does not lead to conflict with the a-theorem, since the trX^9 superpotential is irrelevant for $x < x_{\rm int}(8,3)$, and so the RG flow in question does not exist in that regime.

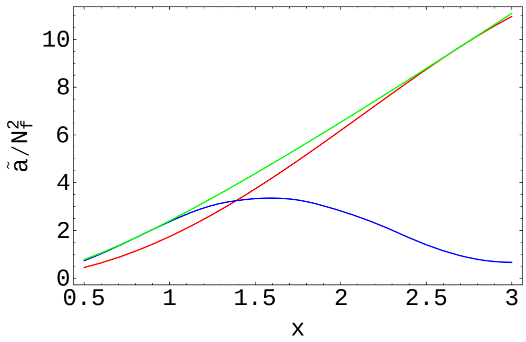

Figure 10: The central charges associated with the infrared fixed point of adjoint SQCD with vanishing superpotential (green), the fixed point **8** associated with superpotential tr X^9 (red), and the fixed point **3** associated with tr X^4 (blue).

All other examples of pairs k, k' that we checked, behave in a qualitatively similar way.

4. Strong-weak coupling duality

In sections 2 and 3 we used the assumption that the U$(1)_R$ current that becomes part of the superconformal algebra at non-trivial fixed points of the RG is visible as an anomaly free symmetry throughout the RG flow. As we discussed in the introduction, this assumption may fail at strong coupling. There is no known a priori way to determine when that will happen, but one way to discover that it does is to use Seiberg duality.

It is thus natural to wonder whether the results of sections 2 and 3 should be modified at strong coupling. For the case of adjoint SQCD with vanishing superpotential, discussed in section 2, there is no known Seiberg-type duality, and so no tools for addressing this question at present. As we saw in sections 2, 3, and will see further in this section, one gets a consistent picture by assuming that no modifications of the results in section 2 are necessary, but that of course does not imply that they are correct for all N_f. It cannot be excluded that for $N_f < N_f^*$ (with some $N_f^* > 0$) the formulae of section 2 are no longer valid, but we have not found any evidence for this in our work.

In this section, we will discuss the fixed point **k** obtained by perturbing adjoint SQCD by the superpotential (3.9) (for $x > x_k$ (3.11)), where a dual description is known to exist [16]–[18], and one can ask what it predicts for the properties of the fixed point **k** at strong coupling.

The duality of [16]–[18] relates adjoint SQCD with gauge group $\mathrm{SU}(N_c)$ and superpotential (3.9) to an $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $\mathrm{SU}(\tilde{N}_c) = \mathrm{SU}(kN_f - N_c)$ and the following matter content: an adjoint field Y, N_f chiral superfields q_i, $\tilde{q}^{\tilde{i}}$ in the anti-fundamental and fundamental representation of the gauge group, respectively, and gauge singlets $(M_j)^i_{\tilde{i}}$, $j = 1, \ldots, k-1$. The superpotential of this theory is given by

$$W_{\mathrm{mag}} = -g'_k \operatorname{tr} Y^{k+1} + \frac{1}{\mu^2} g'_k \sum_{j=1}^{k} M_j \tilde{q} Y^{k-j} q \,, \tag{4.1}$$

where μ is an auxiliary scale. We will refer to these two theories as electric and magnetic, respectively. The conjecture of [16]–[18] is that they flow in the infrared to the same fixed point. The operator matching between the electric and magnetic theories is:[8]

$$\begin{aligned}
&(\mathcal{M}_j)^i_{\tilde{i}} = \tilde{Q}_{\tilde{i}} X^{j-1} Q^i \longleftrightarrow (M_j)^i_{\tilde{i}} \,, \\
&\operatorname{Tr} X^j \longleftrightarrow \operatorname{Tr} Y^j \,, \\
&\mathcal{B}_{el}^{(n_1, n_2, \cdot, n_k)} \longleftrightarrow \mathcal{B}_{mag}^{(m_1, m_2, \cdot, m_k)} \,; \quad m_l = N_f - n_{k+1-l} \,; \quad l = 1, 2, \ldots k \,,
\end{aligned} \tag{4.2}$$

where the magnetic baryons are defined in the same way as the electric ones (3.21) (with the substitution $N_c \to \tilde{N}_c = kN_f - N_c$).

In the next subsection we briefly discuss the magnetic theory from the point of view of the analysis of sections 2, 3. In subsection 4.2 we discuss the implications of its properties for the electric theory.

4.1 RG flows in the magnetic theory

It will be convenient to introduce the magnetic dual of x,

$$\tilde{x} \equiv \frac{\tilde{N}_c}{N_f} = k - x \,. \tag{4.3}$$

We will mainly discuss the region

$$x, \tilde{x} \in \left(\frac{1}{2}, k - \frac{1}{2} \right) \,, \tag{4.4}$$

in which both the electric and the magnetic theories are asymptotically free.

When $\tilde{x}$ is close to (and above) $1/2$, most of the terms in the superpotential (4.1) are irrelevant. The only exception is the term $M_k \tilde{q} q$, which is relevant in the infrared fixed

[8] We neglect numerical proportionality constants.

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point of the magnetic adjoint SQCD for all $\tilde{x} > 1/2$. This term drives the theory to a new fixed point, at which $M_1, \ldots, M_{k-1}$ are free but M_k is interacting. The R-charges and central charge $\tilde{a}^m$ at this fixed point can be determined in a similar way to that employed in section 2. Denoting the R-charge of $q, \tilde{q}$ by $\tilde{y}$, one has $R(Y) = (1-\tilde{y})/\tilde{x}$ and $R(M_k) = 2 - 2\tilde{y}$. Plugging these charges into the expression for $\tilde{a}$, and maximizing w.r.t. $\tilde{y}$, one can determine $\tilde{y}$ and $\tilde{a}^m$.

When $\tilde{x}$ increases further, more and more of the terms in the superpotential (4.1) become relevant and have to be taken into account. To solve for the R-charges, one has to discuss separately two different ranges of $\tilde{x}$:

1. The $\mathrm{tr}Y^{k+1}$ term in (4.1) is irrelevant in the infrared fixed point of the magnetic adjoint SQCD. The last p meson fields, M_j with $j = k-p+1, \ldots, k$ are interacting, while the rest of the meson fields are free. In this case, the R-charge of the interacting mesons is given by

$$R(M_j) = 2 - 2R(q) - (k-j)R(Y) = 2 - 2\tilde{y} - (k-j)\frac{1-\tilde{y}}{\tilde{x}}\,. \tag{4.5}$$

   $\tilde{y}$ is determined by computing the magnetic central charge as a function of $\tilde{y}$ and maximizing it.

2. The $\mathrm{tr}\,Y^{k+1}$ term in the magnetic superpotential is relevant. In this case we set $R_k(Y) = 2/(k+1)$, $y_k = 1 - 2\tilde{x}/(k+1)$, and determine the R-charges of the mesons by using (4.5). If (4.5) gives an R-charge smaller than 2/3 to a meson, the corresponding term in the magnetic superpotential is irrelevant in the infrared, and this meson remains free there.

In practice, one proceeds in a way similar to that employed in sections 2 and 3. Introduce an auxiliary quantity $\tilde{a}^{m,(p)}$, the magnetic central charge computed with the assumption that the last $p = (0, 1, 2, \ldots)$ meson fields $M_k, \ldots, M_{k-p+1}$ *are not* free; their R-charges are given by (4.5).

$$\begin{aligned}\frac{\tilde{a}^{m,(p)}}{N_f^2} &= 6\tilde{x}(\tilde{y}-1)^3 - 2\tilde{x}(\tilde{y}-1) + 3\tilde{x}^2\left(\frac{1-\tilde{y}}{\tilde{x}} - 1\right)^3 - \tilde{x}^2\left(\frac{1-\tilde{y}}{\tilde{x}} - 1\right) + 2\tilde{x}^2 + \\ &\quad + \frac{1}{9}\sum_{j=1}^{p}\left[2 - 3\left(2\tilde{y} + (j-1)\frac{1-\tilde{y}}{\tilde{x}}\right)\right]^2\left[5 - 3\left(2\tilde{y} + (j-1)\frac{1-\tilde{y}}{\tilde{x}}\right)\right] + \frac{2}{9}(k-2p)\,.\end{aligned} \tag{4.6}$$

Start with $\tilde{a}^{m,(1)}$and maximize it w.r.t. $\tilde{y}$. Denote the value of $\tilde{y}$ at the maximum by $\tilde{y}^{(1)}(x)$. Vary $\tilde{x}$ to the point where the R-charge of M_{k-1} (4.5) approaches 2/3. At that point the term $M_{k-1}\tilde{q}Yq$ in the magnetic superpotential (4.1) becomes relevant, and one should switch to the $\tilde{a}^{m,(2)}$ description. This can be continued to arbitrarily large $\tilde{x}$.

The above procedure is valid as long as the polynomial superpotential for Y is irrelevant in the infrared. Define $\tilde{x}_k$ via the condition

$$R(Y^{k+1}) = 2\,, \qquad \text{at } \tilde{x} = \tilde{x}_k\,. \tag{4.7}$$

For $\tilde{x} > \tilde{x}_k$, one should set the R-charges to the values given in point (2) above.

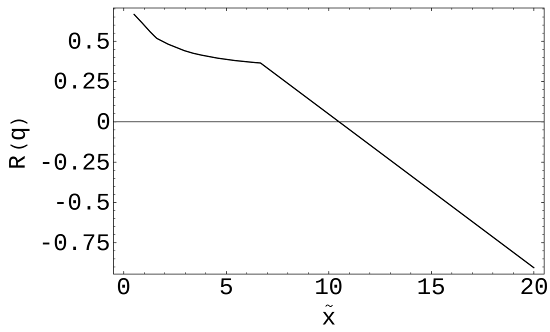

Figure 11: R-charge of q as a function of $\tilde{x}$ at $k = 20$.

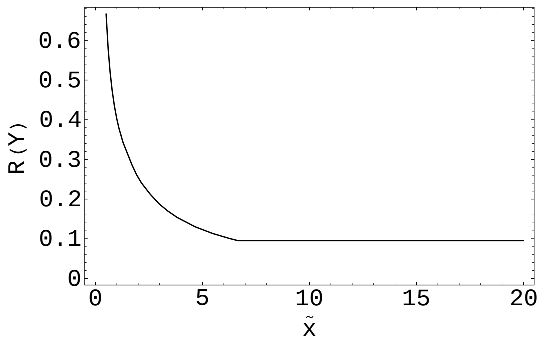

Figure 12: R-charge of Y as a function of $\tilde{x}$ at $k = 20$.

A few comments about the above procedure:

(a) The expression (4.6) looks very similar to the analogous expression in the electric theory (2.13). In particular, the value of $\tilde{y}$ at the maximum for a given p, $\tilde{y}^{(p)}(x)$ can be obtained from the one found in section 2 by replacing $x \to \tilde{x}$ and $y \to \tilde{y}$. Nevertheless, the magnetic central charge $\tilde{a}^m$ is not related to the electric one in the same way. The reason is that in the electric theory we saw in section 2 that one switches from the $\tilde{a}^{(p)}$ to the $\tilde{a}^{(p+1)}$ description when

$$2y + (p-1)\frac{1-y}{x} = \frac{2}{3}, \tag{4.8}$$

while in the magnetic theory the analogous condition is

$$2\tilde{y} + (p-1)\frac{1-\tilde{y}}{\tilde{x}} = \frac{4}{3}. \tag{4.9}$$

(b) One may wonder whether (4.6) should be further corrected to take into account the decoupling of some other chiral superfields. It is easy to see that the answer is no. The operators $\mathrm{tr}Y^j$ can be neglected for the same reasons as in the electric theory (their contribution to $\tilde{a}^{(m)}$ is down by a factor of N_f^2 from the leading terms). The baryons have a large positive R-charge, as in the discussion of sections 2, 3. Finally, the operators $\tilde{q}Y^{j-1}q$ are not chiral, due to the superpotential (4.1).

- (c) In the same way as in section 3, one can derive a bound analogous to (3.19):

$$\frac{\tilde{x}_k}{k+1} < \frac{1-\tilde{y}_{as}}{2}, \tag{4.10}$$

  where $\tilde{y}_{as}$ is computed in appendix A.

The results of our numerical implementation of the above procedure are depicted in figures 11–14. Figures 11 and 12 are the plots of the R-charges of q and Y as a function of $\tilde{x}$ at $k = 20$. For $\tilde{x} < \tilde{x}_k$ these R-charges are determined via the maximization procedure, while for $\tilde{x} > \tilde{x}_k$ they are determined by the superpotential (4.1). Figure 13 exhibits the

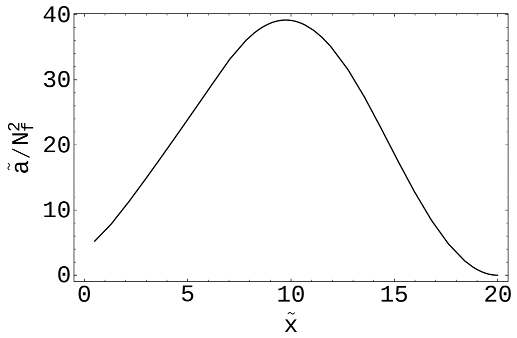

Figure 13: $\tilde{a}_k^m$ as a function of $\tilde{x}$ at $k = 20$.

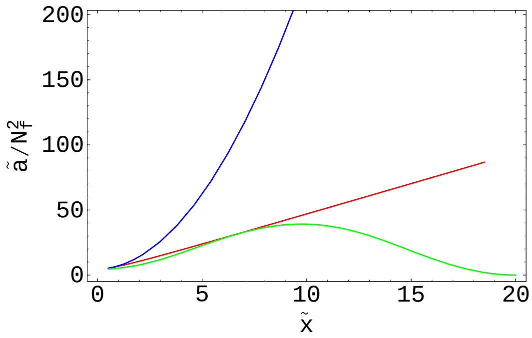

Figure 14: The central charge as a function of $\tilde{x}$ corresponding to the UV (top, blue curve) and IR (middle, red curve) fixed points of the magnetic theory with the coupling in front of $\mathrm{Tr}\, Y^{k+1}$ tuned to zero, and that of the fixed point with the superpotential (4.1) turned on (bottom, green curve), for $k = 20$.

magnetic central charge $\tilde{a}^m$ as a function of $\tilde{x}$. Finally figure 14 is the analog of figure 7 for the magnetic case. It shows the UV central charge, $\tilde{a}^m_{UV}$, which is given by

$$\tilde{a}^m_{UV} = N_f^2 \left[\frac{4}{9}\tilde{x} + \frac{20}{9}\tilde{x}^2 + \frac{2k}{9} \right], \tag{4.11}$$

the infrared central charge of the theory corresponding to the superpotential (4.1) with the $\mathrm{Tr}\, Y^{k+1}$ turned off, and the central charge $\tilde{a}^m_k$ corresponding to the full superpotential with $k = 20$. As in the electric case, the bottom curve in figure 14 should only be taken seriously to the right of the point $\tilde{x}_k$, where it touches the curve above it.

4.2 Consequences for duality

One consequence of the discussion above for the duality of [16]–[18] is that for all k, there exists an analog of the "conformal window" of supersymmetric QCD [15], where both the electric and magnetic theories are asymptotically free. In our case the analogous statement is that there exists a region in x, in which both the electric and the magnetic polynomial superpotentials $\mathrm{tr}X^{k+1}$ and $\mathrm{tr}Y^{k+1}$ are relevant in the infrared. Indeed, in section 3 we saw that $\mathrm{tr}X^{k+1}$ is relevant for $x > x_k$ where $x_k < (k+1)/2$ (see eq. (3.19)). Similarly, the magnetic superpotential is relevant for $\tilde{x} > \tilde{x}_k$ where $\tilde{x}_k < (k+1)/2$ (see eq. (4.10)). Taking into account (4.3), we see that there is a window,

$$x_k < x < k - \tilde{x}_k\,, \tag{4.12}$$

in which both polynomial superpotentials are relevant.

If x_k and $\tilde{x}_k$ instead satisfied the inequality $x_k + \tilde{x}_k > k$, so that the region (4.12) did not exist, the situation would have been much more puzzling. For $k - \tilde{x}_k < x < x_k$, the duality of [16]–[18] would have then predicted an equivalence of theories in which the polynomial superpotentials are turned off on both sides. Such a duality would have had a number of puzzling features, and we view the existence of the "conformal window" (4.12) as a consistency check on the whole picture.

The duality of [16]–[18] implies in particular that the electric and magnetic R-charges and central charges agree for all x,

$$\begin{aligned} R_k(X;x) &= \tilde{R}_k(Y;k-x)\,; \\ \tilde{a}_k^{el}(x) &= \tilde{a}_k^{m}(k-x)\,. \end{aligned} \tag{4.13}$$

It is natural to ask whether equation (4.13) actually agrees with the computations performed in sections 2–4. The answer is that the two calculations agree in the conformal window (4.12), but disagree outside of it. The agreement in the conformal window essentially follows from the anomaly matching that was checked to hold in the original papers [16]–[18].

The disagreement outside the conformal window can be understood even without detailed calculations. It is clear that the flavor of the calculation is completely different on the two sides. Consider, for example the region $x < x_k$. The electric superpotential (3.9) is irrelevant and can be neglected in this region, and thus the R-charges and central charges should be computed as in section 2, by maximizing the trial R-charge (2.2). In the magnetic theory, the polynomial superpotential is strongly relevant in this regime, and thus naively one would expect that no maximization is necessary, and one just uses $\tilde{R}_k(Y) = 2/(k+1)$, etc. It would be very surprising if the maximization process of section 2 gave such a simple, x independent result, and as we saw in section 2, it in fact does not.

A natural interpretation of this disagreement is that, just like in supersymmetric QCD, for $x < x_k$ the magnetic theory is so strongly coupled that the infrared R-charge cannot be identified with any symmetry of the UV theory, and the only way to find the correct answer is to pass to the dual, electric variables. Similarly, the electric description breaks down for $x > k - \tilde{x}_k$ and in that region, one has to use the magnetic variables in order to compute $\tilde{a}_k$. One can check that doing that leads to results consistent with the a-theorem.

5. Discussion

The main motivation for this paper was the recent work of Intriligator and Wecht [1], who proposed a way to determine the $\mathrm{U}(1)_R$ symmetry which belongs to the $\mathcal{N} = 1$ superconformal algebra at an IR fixed point of a SUSY gauge theory. Our purpose was to explore in a specific class of models the interplay between three circles of ideas:

1. The a-theorem: the conjecture that the combination of 't Hooft anomalies (1.1) is always lower at an IR fixed point of the renormalization group than at the corresponding UV fixed point [2]–[11].

2. Seiberg duality: the conjecture that $\mathcal{N} = 1$ supersymmetric gauge theories often have the property that two different theories flow in the infrared to the same fixed point [15]. In the models discussed here, the relevant version of this duality was proposed in [16]–[18].

3. The results of [1] on determining the R-charge at an infrared fixed point of an $\mathcal{N} = 1$ gauge theory.

We showed that the results of [1] (slightly corrected to take into account unitarity constraints) lead, in the class of models that we studied, to a more detailed understanding of the phase structure. The resulting phase diagram provides some rather non-trivial checks of the a-theorem.

As discussed in [1], the a-theorem is guaranteed to hold when using their results, if the UV and IR fixed points are sufficiently close to each other. The checks performed here are non-trivial since many of them are performed in the opposite regime, where the UV and IR fixed points are very far from each other. In particular, we showed that the results of [1] and this paper resolve certain potential problems with the a-theorem raised in [8]. We view the consistency of our results with the a-theorem as evidence for the validity of both.

We also showed that the results of [1] satisfy some non-trivial consistency conditions with the strong-weak coupling duality of [16]–[18]. For example, the fact that there exists a region in (N_f, N_c) where both the electric superpotential $\mathrm{tr}X^{k+1}$, and the magnetic one $\mathrm{tr}Y^{k+1}$ correspond to relevant perturbations of the IR fixed points of the corresponding gauge theories was necessary for the consistency of [16]–[18]; the results of [1] and this paper show that such a region indeed exists. This, too, validates both circles of ideas.

The construction of [1] is useful when one can identify the $\mathrm{U}(1)_R$ symmetry of the IR $\mathcal{N}=1$ superconformal field theory among the anomaly free symmetries of the full theory. We showed that this is usually the case in a finite region in parameter space (e.g. here the space labeled by x (2.1)), and gave examples in which this idea fails. In the examples of this failure discussed in this paper, one can find the right $\mathrm{U}(1)_R$ symmetry by switching to a weakly coupled (Seiberg) dual. It is tempting to conjecture that this is a general feature.

It might be interesting to repeat the analysis done here in other $\mathcal{N}=1$ supersymmetric gauge theories, to see whether any difficulties with the a-theorem and/or Seiberg duality arise. The main open problem related to the subject of this paper is to prove the a-theorem.

Acknowledgments

We thank K. Intriligator and A. Schwimmer for discussions. This work was supported in part by DOE grant #DE-FG02-90ER40560.

A. Large x behavior of the R-charges and the central charge $\tilde{a}$

We start by presenting an analytic solution for $\tilde{a}(x)$ and the R-charges $R(Q)$ and $R(X)$, for the system studied in section 2, in the large x limit defined in (2.1), i.e. for $N_f \ll N_c$, with both $N_f, N_c \to \infty$. To do that, we need to implement the procedure described after eq. (2.13). Since we expect the R-charge of Q, y, to go to a finite constant as $x \to \infty$ (see e.g. figure 2), it is easy to take the $x \to \infty$ limit of (2.13). The first line, which is nothing but $\tilde{a}_0(x,y)$ (2.2), is given in the limit by

$$\tilde{a}_0(x,y) \simeq N_f^2 \left[6x(y-1)^3 - 10x(y-1)\right] . \tag{A.1}$$

Note, in particular, that the terms that go like x^2 in (2.2) cancel, and the leading behavior is linear in x.

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The second line of (2.13) contains a sum over a large number of terms, since p (or more precisely p_0) diverges in the limit $x \to \infty$. Indeed, to estimate p, one can plug into (2.12), and require that $R(\mathcal{M}_p) = 2/3$. The mistake one makes by doing that is subleading in the $1/x$ expansion. This leads to

$$2y + (p-1)\frac{1-y}{x} = \frac{2}{3}\,. \tag{A.2}$$

We see that at large x, p is expected to be proportional to x. To perform the sum on the second line of (2.13), one can replace j by the variable t defined by

$$t = 2y + (j-1)\frac{1-y}{x}\,. \tag{A.3}$$

At large x, t becomes a continuous variable, and one can replace the sum over j by an integral over t. Performing this integral leads to:

$$\frac{1}{9}\sum_{j=1}^{p}\left[2-3\left(2y+(j-1)\frac{1-y}{x}\right)\right]^2\left[5-3\left(2y+(j-1)\frac{1-y}{x}\right)\right] \simeq \frac{N_f^2 x}{18}(2-6y)^3\,. \tag{A.4}$$

Adding this to (A.1), one finds

$$\tilde{a}(x,y) \simeq N_f^2 x\left[6(y-1)^3 - 10(y-1) + \frac{1}{18}(2-6y)^3\right]. \tag{A.5}$$

The local maximum of (A.5) w.r.t. y occurs at

$$y = \frac{\sqrt{3}-1}{3}\,, \tag{A.6}$$

in agreement with the numerical results that lead to figure 2. The value of the central charge $\tilde{a}$ in this limit is

$$\tilde{a} = \frac{4N_f^2 x}{3}(2+\sqrt{3})\,. \tag{A.7}$$

We next move on to the magnetic theory described in section 4. To find an analytic expression for $\tilde{a}^m$, $R(q)$, $R(Y)$, as $\tilde{x} \to \infty$, $\tilde{x} < \tilde{x}_k$, we should implement the procedure outlined after (4.6). Again as in the case above we expect that the R-charge of q, $\tilde{y}$, goes to a finite constant as $\tilde{x} \to \infty$, so it is easy to take a large $\tilde{x}$ limit of (4.6). The first line of (4.6) behaves exactly in the same way as the first line in (2.13) and hence is given by (A.1) (with the replacement $x \to \tilde{x}$ and $y \to \tilde{y}$) in the large $\tilde{x}$ limit.

The second line of (4.6) contains a sum over a large number of terms, and a term linear in p: $\frac{2N_f^2}{9}(k-2p)$. We can use (4.9) to estimate the value of p

$$p = \left(\frac{4}{3} - 2\tilde{y}\right)\frac{\tilde{x}}{1-\tilde{y}} + \mathcal{O}(1)\,. \tag{A.8}$$

To perform the sum we introduce the variable $\tilde{t}$ defined by

$$\tilde{t} = 2\tilde{y} + (j-1)\frac{1-\tilde{y}}{x}\,. \tag{A.9}$$

At large $\tilde{x}$, $\tilde{t}$ becomes a continuos variable and one can replace the sum over j by an integral over $\tilde{t}$. The main difference with the case discussed above is that the limits of integration are different. While the integral over t is performed over the interval $(2y, \frac{2}{3})$, the $\tilde{t}$ integral runs over the interval $(2y, \frac{4}{3})$. Performing the integral we get

$$\frac{1}{9}\sum_{j=1}^{p}\left[2-3\left(2\tilde{y}+(j-1)\frac{1-\tilde{y}}{\tilde{x}}\right)\right]^2\left[5-3\left(2\tilde{y}+(j-1)\frac{1-\tilde{y}}{\tilde{x}}\right)\right] \simeq \frac{N_f^2\tilde{x}}{18}(2-6\tilde{y})^3 + + \frac{4N_f^2\tilde{x}}{27(1-\tilde{y})}. \tag{A.10}$$

Adding all the pieces we obtain the following expression for $\tilde{a}^m$

$$\tilde{a}^m = N_f^2\tilde{x}\left[6(\tilde{y}-1)^3 - 10(\tilde{y}-1) + \frac{1}{18}(2-6\tilde{y})^3 + \frac{4(2\tilde{y}-1)}{9(1-\tilde{y})}\right] + \frac{2k}{9}N_f^2\,. \tag{A.11}$$

The local minimum of this expression with respect to $\tilde{y}$ occurs at

$$\tilde{y}_{as} \simeq 0.2844\,. \tag{A.12}$$

The value of central charge in this limit is

$$\tilde{a}^m \simeq 4.6909N_f^2\tilde{x} + \frac{2}{9}kN_f^2\,. \tag{A.13}$$

Finally the point $\tilde{x}_k$, where the coupling (4.1) becomes relevant is

$$\tilde{x}_k \simeq 0.3578k\,. \tag{A.14}$$

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